Geometrical approach to Seidel's switching for strongly regular graphs

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Abstract

In this paper, we simplify the known switching theorem due to Bose and Shrikhande as follows. Let $G = (V, E)$ be a primitive strongly regular graph with parameters (v, k, λ, μ). Let $S(G, H)$ be the graph from G by switching with respect to a nonempty $H \subset V$. Suppose $v = 2(k - \theta_1)$ where θ_1 is the nontrivial positive eigenvalue of the $(0, 1)$ adjacency matrix of G. This strongly regular graph is associated with a regular two-graph. Then, $S(G, H)$ is a strongly regular graph with the same parameters if and only if the subgraph induced by H is $k - \frac{v-h}{2}$ regular. Moreover, $S(G, H)$ is a strongly regular graph with the other parameters if and only if the subgraph induced by H is $k - \mu$ regular and the size of H is $v/2$. We prove these theorems with the view point of the geometrical theory of the finite set on the Euclidean unit sphere.

1 Introduction

A simple graph $G = (V, E)$ is called a strongly regular graph with parameters (v, k, λ, μ) if the cardinality of V is v, G is k regular, any two adjacent vertices are adjacent to λ common vertices, and any two nonadjacent vertices are adjacent to μ common vertices. The complement of a strongly regular graph is also strongly regular. A strongly regular graph is said to be primitive if both it and its complement are connected. It is known that an imprimitive strongly regular graph is either a complete multipartite graph, or the disjoint union of a number of copies of a complete graph. Primitive strongly regular graphs are known as association schemes of class 2, or distance regular graphs of diameter 2. The $(0, 1)$ adjacency matrix of a graph G is defined by the matrix indexed by the vertices, whose (x, y) entry is 1 if x is adjacent to y, and 0 otherwise. Let A_1 be a $(0, 1)$ adjacency matrix of a strongly regular graph, and A_2 be that of the complement. Then, the identity matrix I, A_1 and A_2 generate the commutative algebra, called the Bose-Mesner algebra. Let E_i $(i = 0, 1, 2)$ be the primitive idempotents of the Bose-Mesner algebra, where $E_0 := J/v$, $E_i E_j = \delta_{i,j} E_i$, J is the all one matrix, and $\delta_{i,j}$ is the Kronecker's delta. We can write A_i and E_i as linear combinations of each other, namely $A_i = \sum_{i=0}^{2} p_i(j) E_j$ and $E_i = \frac{1}{v} \sum_{i=0}^{j} q_i(j) A_j$. $P = (p_i(j))$, whose $(j+1, i+1)$ entry is $p_i(j)$, and $Q = (q_i(j))$, whose $(j+1, i+1)$ entry is $q_i(j)$, are called the first and second eigenmatrices, respectively. P and Q give a lot of information of the strongly regular graph, but both of them depend only on parameters (v, k, λ, μ). However, even if two strongly regular graphs have the same parameters, there is a possibility that they are not isomorphic to each other. For examples, we have the exactly four strongly regular graphs with parameters $(28, 12, 6, 4)$, those are the triangular graph $T(8)$ and three other graphs called Chang graphs [9].

By Seidel's switching of edges of a strongly regular graph associated with a regular two-graph, we may get new strongly regular graphs with the same parameters. Bose and Shrikhande determine the conditions of the switching set which may give a new strongly regular graph [2]. In this paper, we simplify the conditions of the switching set. Let $S(G, H)$ denote the graph from G by switching with respect to $H \subset V$. Let $k > \theta_1 > \theta_2$ denote the eigenvalues of $(0, 1)$ adjacency matrix of a strongly regular graph. The following are the main theorems in this paper.

Theorem 1.1. *Let $G = (V, E)$ be a primitive strongly regular graph with parameters (v, k, λ, μ). H is a subset of V, and its cardinality is h. Suppose $v = 2(k - \theta_1)$. Then, the following are equivalent:*
  (i) *$S(G, H)$ is a strongly regular graph with parameters (v, k, λ, μ).*
  (ii) *The subgraph induced by H is $k - \frac{v-h}{2}$ regular.*

Theorem 1.2. *Let $G = (V, E)$ be a primitive strongly regular graph with the parameters (v, k, λ, μ). H is a subset of V. Suppose $v = 2(k - \theta_1)$. Then, the following are equivalent.*
  (i) *$S(G, H)$ is a strongly regular with parameters $(v, k + c, \lambda + c, \mu + c)$ where $c = v/2 - 2\mu$.*
  (ii) *The cardinality of H is equal to $v/2$ and the subgraph induced by H is $k - \mu$ regular.*

For examples, the complements of $T(8)$ or the Chang graphs are able to apply Theorem 1.1. In particular, by Theorem 1.1, we can find at least 100000 strongly regular graphs with parameters $(276, 140, 58, 84)$. It is known that there are at least 7715 strongly regular graphs with these parameters [12]. We prove these theorems with the view point of the geometrical theory of the finite subset on the Euclidean unit sphere.

2 Preliminaries

In this section, we introduce some basic terminology and results. More details may be found, for examples, in [1], [4], [10], [14], [16] and [18].

2.1 Regular two-graphs

Let V be a set of vertices, and Δ be a collection of 3-subsets of V, where an n-subset means a subset whose cardinality is n. (V, Δ) is called a two-graph if each 4-subset of V contains an even number of elements of Δ. A two-graph (V, Δ) is said to be regular if each 2-subset of V is contained in a constant number of Δ.

Given a simple graph $G = (V, E)$, the set Δ of 3-subsets of V whose induced subgraph in G contains an odd number of edges give rise to a two-graph (V, Δ). In fact, every two-graph can be represented in this way. Switching from G with respect to a subset $H \subset V$ consists of interchanging adjacency and non-adjacency between H and its complement $V \setminus H$. Graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ represent the same two-graph if they are related by the equivalence relation of switching with respect to some $H \subset V$. An equivalence class of graphs under switching is called a switching class. Thus, a two-graph can be identified with the switching class of a graph.

The $(0, -1, 1)$ adjacency matrix B of a graph G is defined by the matrix indexed by the vertices, whose (x, y) entry is -1 if x is adjacent to y, 1 if x is not adjacent to y, and 0 if $x = y$. The $(0, -1, 1)$ adjacency matrix of $S(G, H)$ is $D_H B D_H$, where B is the $(0, -1, 1)$ adjacency matrix of G and D_H is the diagonal matrix whose (x, x) entry is -1 for $x \in H$ and 1 for $x \in V \setminus H$. Because B and $D_H B D_H$ have the same eigenvalues, the eigenvalues of a two-graph are the eigenvalues of the $(0, -1, 1)$ adjacency matrix of any graph in its switching class. A two-graph (V, Δ) is regular if and only if it has two distinct eigenvalues $\rho_1 > 0 > \rho_2$, where $\rho_1 \rho_2 = 1 - |V|$.

The switching class of a graph G, and hence any two-graph, can be represented geometrically as a set of equiangular lines. Let $-\rho < 0$ be the smallest eigenvalue of the $(0, -1, 1)$ adjacency matrix B of G on v vertices, and suppose that $-\rho$ has multiplicity $v - d$. Then $\rho I + B$ is positive semidefinite of rank d and so can be represented as the Gram matrix of the inner products of n vectors in Euclidean space $\mathbb{R}^d$, which implies the equiangular line having the same angle ϕ with $\cos \phi = 1/\rho$. Here, the Gram matrix of a finite set $X \subset \mathbb{R}^d$ is indexed by X, and its (x, y) entry is the usual inner product of x and y. Conversely, given a set of v nonorthogonal equiangular lines in $\mathbb{R}^d$, there exists a two-graph from which

it can be constructed by this method. The cardinality v of such a set satisfies $v \leq d(\rho^2 - 1)/(\rho^2 - d)$, and this bound is achieved if and only if the corresponding two-graph is regular.

The matrix $I + \frac{1}{\rho}B$ is the Gram matrix of a finite set X on S^{d-1}. Similarly, we can get the finite set $X_H \subset S^{d-1}$ with the Gram matrix $I + \frac{1}{\rho}D_H B D_H$ from $S(G, H)$. We define the bijection $\varphi : V \to X$. The switching with respect to H means that we move $\varphi(H)$ to the antipodal part $-\varphi(H)$ in the spherical embedding. Namely, we have $X_H = (X \setminus \varphi(H)) \cup (-\varphi(H)) = \{x \in X \mid x \notin \varphi(H)\} \cup \{-x \mid x \in \varphi(H)\}$.

2.2 Embedding to the unit sphere

Let $G = (V, E)$ be a primitive strongly regular graph. A primitive strongly regular graph is identified with a symmetric association scheme $(V, \{R_0, R_1, R_2\})$ with two classes, where $R_0 = \{(x, x) \mid x \in V\}$ and $R_1 = \{(x, y) \mid (x, y) \in E\}$. Let A_i be the $(0, 1)$ adjacency matrix with respect to the relation R_i, E_i be the primitive idempotents, and m_i be the rank of E_i. As is well known, the spherical embedding of V with respect to E_i $(i = 1, 2)$ in the unit sphere $S^{m_i - 1}$ are defined as follows. We identify $x \in V$ with the vectors $\bar{x} = \sqrt{\frac{|V|}{m_i}} E_i e_x$, where $e_x = {}^t(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^v$ with the x-th coordinate 1. If the strongly regular graph is primitive, then this embedding is faithful. The standard inner product $\langle \bar{x}, \bar{y} \rangle$ in $\mathbb{R}^{m_i}$ is given by $q_i(j)/m_i = p_j(i)/k_j$ if $(x, y) \in R_j$, where $p_j(i)$ and $q_i(j)$ are entries of the first and second eigenmatrix, $m_i = q_i(0)$ and $k_j = p_j(0)$. Namely, this spherical embedding has the structure of the strongly regular graph. We know the properties of this embedding as s-distance sets and spherical t-designs.

We introduce the concept of s-distance sets and spherical t-designs. Let X be a nonempty finite subset of S^{d-1}. Define $A(X) := \{\langle x, y \rangle \mid x, y \in X, x \neq y\}$, that is, the set of the standard inner products of distinct vectors of X. X is called an s-distance set if $|A(X)| = s$. X is called a spherical t-design on S^{d-1}, if $\sum_{x \in X} f(x) = 0$ for any $f \in \mathrm{Harm}_l(\mathbb{R}^d)$ with $1 \leq l \leq t$, where $\mathrm{Harm}_l(\mathbb{R}^d)$ is the linear space of harmonic homogeneous polynomials of degree l, with d variables.

If X is an s-distance set and a spherical t-design, and $t \geq 2s - 2$, then $(X, \{R_i\})$ is an association scheme of class s, where $R_i = \{(x, y) \in X \times X \mid \langle x, y \rangle = \alpha_i\}$, and $A(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ [10]. In particular, X is a 2-distance set and a spherical 2-design, then X has the structure of a strongly regular graph.

For a fixed $x \in X \subset S^{d-1}$, we define $n_i(x) := |\{y \in X \mid \langle x, y \rangle = \alpha_i\}|$. If $t \geq s - 1$, then X is distance invariant, that is, $n_i(x)$ is a constant number k_i for any $x \in X$ [10].

Let φ_i be the embedding bijection from the vertex set of a primitive strongly regular graph $G = (V, E)$ to $S^{m_i - 1}$ with respect to E_i $(i = 1, 2)$. $\frac{|V|}{m_i} E_i$ is the Gram matrix of the spherical embedding because ${}^t E_i E_i = E_i$. Since we can write $E_i := \frac{1}{|V|} \sum_{j=0}^{2} q_i(j) A_j$, $\varphi(V)$ is a 2-distance set. Moreover, it is known that $\varphi_i(V)$ is a spherical 2-design on $S^{m_i - 1}$ [8].

3 Known switching theorems

Bose and Shrikhande proved the following theorem in 1970.

Theorem 3.1 (Theorem 8.1 in [2]). *Let $G = (V, E)$ be a strongly regular graph with the parameters (v, k, λ, μ) where $2k - v/2 = \lambda + \mu$. Let H_1 be a subset of V, and $H_2 := V \setminus H_1$. Let v_i be the cardinalities of H_i. Then, the following are equivalent.*
(i) $S(G, H_1)$ is strongly regular.
(ii) The subgraph induced by H_1 is w_1 regular and the subgraph induced by H_2 is w_2 regular where

$$w_1 - w_2 = \frac{v_1 - v_2}{2}.$$

Note that if there exists nonempty H such that $S(G, H)$ is strongly regular, then G has the condition $2k - v/2 = \lambda + \mu$ [2]. In particular, when $S(G, H)$ is strongly regular with the same parameters, we have the following theorem.

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Theorem 3.2 (Theorem 8.3 in [2])**.** *Let $G = (V, E)$ be a strongly regular graph with the parameters (v, k, λ, μ) where $2k - v/2 = \lambda + \mu$. Let H_1 be a subset of V, and $H_2 := V \setminus H_1$. Then, the following are equivalent.*

(i) *$S(G, H_1)$ is strongly regular.*

(ii) *In G each vertex in H_1 is adjacent to exactly half of vertices in H_2, and each vertex in H_2 is adjacent to exactly half of vertices in H_2.*

It is known that G is a strongly regular graph with $2k - v/2 = \lambda + \mu$ if and only if G is a strongly regular graph with $v = 2(k - \theta_1)$ or $v = 2(k - \theta_2)$ [5]. If G has the condition $v = 2(k - \theta_2)$, then the complements $\bar{G}$ has the condition $v = 2(k - \bar{\theta_1})$ where $\bar{\theta_1}$ is the positive eigenvalue of $\bar{G}$. Therefore, without loss of generality, we may assume $v = 2(k - \theta_1)$. Hence, Theorem 1.1 is the simplification of Theorem 3.2. The switching class of a regular two-graph may contain strongly regular graphs with at most two parameters sets [5, 6]. Theorem 1.2 is the simplification of Theorem 3.1 in the case where $S(G, H)$ has the other parameters.

4 Proof of Theorem 1.1

Let $G = (V, E)$ be a primitive strongly regular graph with parameters (v, k, λ, μ). Let k, θ_1 and θ_2 be the eigenvalues of the $(0, 1)$ adjacency matrix of G, where $k > \theta_1 > 0 > \theta_2$. G is identified with an association scheme $(V, \{R_0, R_1, R_2\})$, where A_1 is the $(0, 1)$ adjacency matrix of G. Then, we can write the first eigenmatrix

$$P = \begin{bmatrix} 1 & k & v - 1 - k \\ 1 & \theta_1 & -1 - \theta_1 \\ 1 & \theta_2 & -1 - \theta_2 \end{bmatrix}$$

where

$$\{\theta_1, \theta_2\} = \left\{ \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 - 4(\mu - k)}}{2}, \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 - 4(\mu - k)}}{2} \right\}.$$

Assume $v = 2(k - \theta_1)$. Then, we can determine $\theta_1 = k - v/2$ and $\theta_2 = \lambda - \mu - k + v/2 = k - 2\mu$. Since $\theta_1 > 0$, we have $k > \frac{v}{2}$. Let B be the $(0, -1, 1)$ adjacency matrix of G, and $-\rho < 0$ be the minimum eigenvalues of B. Since $B = -A_1 + A_2$, the eigenvalues of B are $-p_1(j) + p_2(j)$ with $j = 0, 1, 2$. Because we have $-p_1(0) + p_2(0) = k - (v - 1 - k) = -1 - 2\theta_1$, the eigenvalues of B are $-1 - 2\theta_1$ and $-1 - 2\theta_2$. Therefore, this strongly regular graph is obtained in the switching class of a regular two-graph. Then, $-\rho = -1 - 2\theta_1$.

Let φ_2 be the spherical embedding bijection from V to $S^{m_2 - 1}$ with respect to E_2. The following is a key lemma to prove the Theorem 1.1.

Lemma 4.1. *Let $G = (V, E)$ be a strongly regular graph with parameters (v, k, λ, μ), where $v = 2(k - \theta_1)$ (resp. $v = 2(k - \theta_2)$). Let B be the $(0, -1, 1)$ adjacency matrices of G, and $-\rho_1 < 0$ (resp. $\rho_2 > 0$) be the minimum (resp. maximum) eigenvalue of B. Let X be a finite set with the Gram matrix $I + \frac{1}{\rho_1} B$ (resp. $I - \frac{1}{\rho_2} B$). Then, X coincides with the spherical embedding with respect to E_2 (resp. E_1).*

Proof. Suppose $G = (V, E)$ has the condition $v = 2(k - \theta_1)$. Let A_i and E_i be defined above. Note that for $i = 0, 1$,

$$E_i(\rho_1 I + B) = E_i(\rho_1 I - A_1 + A_2) = \rho_1 E_i - p_1(i) E_i + p_2(i) E_i = 0.$$

Therefore, $I + \frac{1}{\rho_1} B$ is equal to $\frac{m_2}{v} E_2$. We can prove the case $v = 2(k - \theta_2)$ in the same manner. $\square$

By Lemma 4.1, the following is clear.

Corollary 4.2. *Let X be the finite set defined in Lemma 4.1. Then, X is a 2-distance set and a spherical 2-design on $S^{m_2 - 1}$ (resp. $S^{m_1 - 1}$), where m_2 (resp. m_1) is the multiplicity of the negative (resp. non trivial positive) eigenvalue of A_1.*

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let $G = (V, E)$ be a primitive strongly regular graph with parameters (v, k, λ, μ), where $v = 2(k - \theta_1)$. Let B be the $(0, -1, 1)$ adjacency matrix of G, $-\rho$ be the minimum eigenvalue of B, and $v - m_2$ is the multiplicity of $-\rho$. Let X be a finite set on S^{m_2-1} with the Gram matrix $I + \frac{1}{\rho}B$. Then, X is a 2-distance set and a spherical 2-design on S^{m_2-1} by Corollary 4.2. Let φ be the spherical embedding bijection $V \to X$ with respect to E_2.

First, suppose $S(G, H) = (V_H, E_H)$ is a strongly regular graph with the same parameters (v, k, λ, μ). Then, the $(0, -1, 1)$ adjacency matrix of $S(G, H)$ is $D_H B D_H$, where D_H is defined above. Let X_H be the finite set on S^{m_2-1} with the Gram matrix $I + \frac{1}{\rho}D_H B D_H$. Similarly, X_H is a 2-distance set and a spherical 2-design on S^{m_2-1}. Note that $X_H = (X \setminus \varphi(H)) \cup (-\varphi(H))$.

Since X is a spherical 2-design, for any $f_1 \in \mathrm{Harm}_1(\mathbb{R}^{m_2})$,

$$0 = \sum_{x \in X} f_1(x) = \sum_{x \in X \setminus \varphi(H)} f_1(x) + \sum_{x \in \varphi(H)} f_1(x). \tag{4.1}$$

On the other hand, for any $f_1 \in \mathrm{Harm}_1(\mathbb{R}^{m_2})$,

$$0 = \sum_{x \in X_H} f_1(x) = \sum_{x \in X \setminus \varphi(H)} f_1(x) + \sum_{x \in -\varphi(H)} f_1(x) = \sum_{x \in X \setminus \varphi(H)} f_1(x) - \sum_{x \in \varphi(H)} f_1(x) \tag{4.2}$$

because X_H is a spherical 2-design and f_1 is a homogeneous polynomial of degree 1. By equations (4.1) and (4.2), we have $\sum_{x \in \varphi(H)} f_1(x) = 0$ for any $f_1 \in \mathrm{Harm}_1(\mathbb{R}^{m_2})$. Therefore, $\varphi(H)$ is a spherical 1-design. Since $\varphi(H)$ is a 1- or 2-distance set and a spherical 1-design, $\varphi(H)$ is distance invariant. Thus, the subgraph induced by H is n regular for some n. It is known that $\varphi(H)$ is a spherical 1-design if and only if $\sum_{x \in \varphi(H)} \sum_{y \in \varphi(H)} \langle x, y \rangle = 0$. Therefore, $1 - \frac{1}{\rho}n + \frac{1}{\rho}(h - n - 1) = 0$, and hence $n = \frac{h-1+\rho}{2} = k - \frac{v-h}{2}$.

Second, suppose the subgraph induced by $H \subset V$ is $k - \frac{v-h}{2}$ regular. Clearly, $\sum_{x \in \varphi(H)} \sum_{y \in \varphi(H)} \langle x, y \rangle = 0$. Therefore, $\varphi(H)$ is a spherical 1-design on S^{m_2-1}.

Let X_H be the finite set on S^{m_2-1} with the Gram matrix $I + \frac{1}{\rho}D_H B D_H$. Then, $X_H := (X \setminus \varphi(H)) \cup (-\varphi(H))$, and X_H has the structure of $S(G, H)$.

For any $f_1 \in \mathrm{Harm}_1(\mathbb{R}^{m_2})$,

$$\sum_{x \in X_H} f_1(x) = \sum_{x \in X \setminus \varphi(H)} f_1(x) + \sum_{x \in -\varphi(H)} f_1(x) = \sum_{x \in X \setminus \varphi(H)} f_1(x) + \sum_{x \in \varphi(H)} f_1(x) = \sum_{x \in X} f_1(x) = 0, \tag{4.3}$$

because $\varphi(H)$ and X are spherical 1-designs. For any $f_2 \in \mathrm{Harm}_2(\mathbb{R}^{m_2})$,

$$\sum_{x \in X_H} f_2(x) = \sum_{x \in X \setminus \varphi(H)} f_2(x) + \sum_{x \in -\varphi(H)} f_2(x) = \sum_{x \in X \setminus \varphi(H)} f_2(x) + \sum_{x \in \varphi(H)} f_2(x) = \sum_{x \in X} f_2(x) = 0 \tag{4.4}$$

because X is a spherical 2-design on S^{m_2-1}, and f_2 is a homogeneous polynomial of degree 2. Therefore, X_H is a spherical 2-design on S^{m_2-1}. Since X_H is a 2-distance set and a spherical 2-design on S^{m_2-1}, $S(G, H)$ is a strongly regular graph. Since X_H is a spherical 1-design and $A(X_H) = A(X)$, $S(G, H)$ is k regular. This implies that $S(G, H)$ has the same parameters (v, k, λ, μ). $\square$

We give some remarks of Theorem 1.1.

Since $k - \frac{v-h}{2}$ is an integer, $h \equiv v \mod 2$.

Suppose $H_1 \subset V$ and $H_2 \subset V$ hold the conditions in Theorem 1.1, (ii). Let S_1 and S_2 be the subgraph induced by H_1 and H_2, respectively. If there exists $g \in \mathrm{Aut}(G)$, such that $S_1{}^g = S_2$, then $S(G, H_1)$ is isomorphic to $S(G, H_2)$.

$S(G, H_1)$ may be isomorphic to $S(G, H_2)$, even when there does not exist $g \in \mathrm{Aut}(G)$, such that $S_1{}^g = S_2$.

We introduce another proof of Theorem 1.1. The author got this proof from a personal communication by A.E. Brouwer [3, 15].

Let $G = (V, E)$ is a strongly regular graph defined in Theorem 1.1.

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First, we suppose $S(G,H)$ is a strongly regular graph with the same parameters. Let A be the $(0,1)$ adjacency matrix of G, which is partitioned according to $\{H, V \setminus H\}$. Namely,

$$A = \begin{bmatrix} A_H & C \\ {}^t C & A_{V \setminus H} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

where $A_H \,(= A_{1,1})$ is the $(0,1)$ adjacency matrix of the subgraph induced by H, and $A_{V \setminus H} \,(= A_{2,2})$ is that by $V \setminus H$. Switching with respect to H implies replacing C to $J - C$ in A. Since $S(G,H)$ is also k regular, the number of entries 1 in C is equal to that in $J - C$. Thus, the number of entries 1 in C is equal to $h(v-h)/2$. Let $f_{i,j}$ denote the average row sum of $A_{i,j}$. Then, $F = (f_{i,j})$ is called the quotient matrix. Since the number of entries 1 in C is $h(v-h)/2$, we can get

$$F = \begin{bmatrix} k - \frac{v-h}{2} & \frac{v-h}{2} \\ \frac{h}{2} & k - \frac{h}{2} \end{bmatrix}.$$

The eigenvalues of F are k and $k - v/2$, because the row sums are k and the trace is $2k - v/2$. It is known that the eigenvalues of A interlace the eigenvalues of F [5]. Namely, $k \geq \theta_1 \geq k - \frac{v}{2} \geq \theta_2$. Since $v = 2(k - \theta_1)$, the interlacing is tight (*i.e.* $\theta_1 = k - v/2$). Therefore, this partition is equitable (*i.e.* the row sum of each $A_{i,j}$ is constant), namely, the subgraph induced by H is $k - (v-h)/2$ regular [5].

Second, suppose the subgraph induced by H is $k - (v-h)/2$ regular. Then, the quotient matrix F is the same above. Hence, the interlacing of eigenvalues of A and F is tight, and hence the partition is equitable. Therefore, $S(G,H)$ is regular, and hence $S(G,H)$ is a strongly regular graph [11]. Moreover $S(G,H)$ has the same parameters as that of G.

5 Proof of Theorem 1.2

The following is a key result in order to prove Theorem 1.2.

Theorem 5.1. *Let $G = (V, E)$ be a strongly regular graph with $v = 2(k - \theta_1)$. If there are $H \subset V$ such that $S(G,H)$ is a strongly regular graph with the other parameters. Then, the spherical embedding with respect to E_2 is on two parallel hyperplanes of dimension at most $m_2 - 1$.*

Proof. Let H be a subset of V. Suppose that $S(G,H)$ is a strongly regular graph with the other parameters. Note that if G has the eigenvalues k, θ_1 and θ_2, then $S(G,H)$ has the eigenvalues k^*, θ_1 and θ_2, where $k^* = k + v/2 - 2\mu$ is degree of $S(G,H)$. Let m_i be the multiplicities of θ_i as eigenvalues of G, and m_i^* be those of $S(G,H)$. Then, $m_1 + 1 = m_2^*$ and $m_2 = m_2^* + 1$. Since $S(G,H)$ is in the switching class of the same regular two-graph as that of G, $S(G,H)$ has the condition $v = 2(k^* - \theta_2)$. By Lemma 4.1, the primitive idempotent of $S(G,H)$ is

$$\frac{v}{m_1^*} E_1^* = I + \frac{1}{\rho^*} A_1^* - \frac{1}{\rho^*} A_2^*$$

where $\rho^* = -1 - 2\theta_2$, and A_1^* and A_2^* are the $(0,1)$-adjacency matrix of $S(G,H)$ and that of the complement, respectively. $D_H E_2 D_H$ is in the Bose-Mesner algebra of $S(G,H)$. Then,

$$\begin{aligned} D_H E_2 D_H + E_1^* &= \frac{m_2}{v}\left(I - \frac{1}{\rho}A_1^* + \frac{1}{\rho}A_2^*\right) + \frac{m_1^*}{v}\left(I + \frac{1}{\rho^*}A_1^* - \frac{1}{\rho^*}A_2^*\right) \\ &= \frac{m_2 + m_1^*}{v}I + \frac{m_1^*\rho - m_2\rho^*}{v\rho\rho^*}(A_1^* - A_2^*) \\ &= \frac{m_2 + m_1 + 1}{v}I + \frac{(m_1+1)(1+2\theta_1) + m_2(1+2\theta_2)}{v\rho\rho^*}(A_1^* - A_2^*) \\ &= I \end{aligned}$$

where $\rho = 1 + 2\theta_1$. Therefore, $D_H E_2 D_H$ is equal to $E_0^* + E_2^*$. Since

$$E_2 D_H j = D_H(E_0^* + E_2^*)j = v D_H j$$

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where j is the all one column vector, the spherical embedding with respect to E_2 is on two hyperplanes which are perpendicular to $D_H j$.

$\square$

The finite set with the Gram matrix $D_H E_2 D_H$ is on one hyperplane of dimension $m_2 - 1$, and is identified with the spherical embedding with respect to E_2^*. Since the spherical embedding X with respect to E_2 is a spherical 1-design, $\sum_{x \in X} x = 0$ and hence the cardinality of the switching set H is equal to $v/2$ by Theorem 5.1.

Proof of Theorem 1.2. First, suppose $S(G, H)$ is strongly regular with the other parameters. Then, the cardinality of H is equal to $v/2$. Let A be the $(0, 1)$ adjacency matrix of G, which is partitioned according to $\{H, V \setminus H\}$. Namely,

$$A = \begin{bmatrix} A_H & C \\ {}^tC & A_{V \setminus H} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

where $A_H \ (= A_{1,1})$ is the $(0, 1)$ adjacency matrix of the subgraph induced by H, and $A_{V \setminus H} \ (= A_{2,2})$ is that by $V \setminus H$. By Theorem 3.1, the both subgraphs induced by H and $V \setminus H$ are n regular for some integer n. The number of entries 1 in C is $v(k - n)/2$. After switching with respect to H, C becomes $J - C$. Therefore, the number of entries 1 in the $(0, 1)$ adjacency matrix of $S(G, H)$ is $v(2n - k + v/2)$. On the other hand, $S(G, H)$ is $k + v/2 - 2\mu$ regular, and the number of entries 1 in the $(0, 1)$ adjacency matrix of $S(G, H)$ is $v(k + v/2 - 2\mu)$. Thus, $n = k - \mu$.

Second, suppose H is $k - \mu$ regular and its cardinality is $v/2$. Let A be the $(0, 1)$ adjacency matrix of G, which is partitioned according to $\{H, V \setminus H\}$. The quotient matrix is

$$F = \begin{bmatrix} k - \mu & \mu \\ \mu & k - \mu \end{bmatrix}.$$

Then, the eigenvalues of F are k and $k - 2\mu = \theta_2$. This interlacing is tight, and hence this partition is equitable. Hence, $S(G, H)$ is a strongly regular graph whose degree $k + v/2 - 2\mu$. $\square$

We introduce another method of determining the cardinality of the switching set H in Theorem 1.2 (ii) [3, 15].

By Theorem 3.1 (ii), the subgraph induced by H_1 is w_1 regular, and hence each vertex H_1 is adjacent to $k - w_1$ vertices of H_2. After switching, each vertex in H_1 is adjacent to w_1 vertices in H_1, and to $v - v_1 - (k - w_1)$ vertices in H_2. Therefore, each vertex in H_1 is adjacent to $v - v_1 - k + 2w_1$ vertices in $S(G, H_1)$. Hence, $k + v/2 - 2\mu = v - v_1 - k + 2w_1$ and

$$w_1 = k - \mu - v/4 + v_1/2. \tag{5.1}$$

Similarly, each vertex in H_2 is adjacent to $v - v_2 - k + 2w_2$ vertices in $S(G, H_1)$ after switching. Since $v_2 = v - v_1$ and $k + v/2 - 2\mu = v - v_2 - k + 2w_2$, we have

$$w_2 = k - \mu + v/4 - v_1/2. \tag{5.2}$$

By counting the number of edges between H_1 and H_2, we have $v_1(k - w_1) = v_2(k - w_2)$. Therefore, by equations (5.1) and (5.2), we get $v_1(\mu + v/4 - v_1/2) = (v - v_1)(\mu - v/4 + v_1/2)$, i.e., $(v_1 - v/2)(v/2 - 2\mu) = 0$. The case $v/2 - 2\mu = 0$ corresponds to $c = 0$. Thus, $v_1 = v/2$.

6 Applications

When $v \leq 280$, the known strongly regular graphs with $v = 2(k - \theta_1)$ have the following parameters [4].

$$\{(v, k, \lambda, \mu)\} = \{(10, 6, 3, 4), (16, 10, 6, 6), (16, 9, 4, 6), (26, 15, 8, 9), (28, 15, 6, 10), (36, 21, 12, 12),$$
$$(36, 20, 10, 12), (50, 28, 15, 16), (64, 36, 20, 20), (64, 35, 18, 20), (82, 45, 24, 25), (100, 55, 30, 30),$$
$$(100, 54, 28, 30), (120, 68, 40, 36), (120, 63, 30, 36), (122, 66, 35, 36), (126, 75, 48, 39),$$
$$(126, 65, 28, 39), (136, 75, 42, 40), (136, 72, 36, 40), (144, 78, 42, 42), (144, 77, 40, 42),$$
$$(170, 91, 48, 49), (176, 105, 68, 54), (176, 90, 38, 54), (196, 104, 54, 56), (210, 110, 55, 60),$$
$$(226, 120, 63, 64), (256, 136, 72, 72), (256, 135, 70, 72), (276, 140, 58, 84), (280, 144, 68, 80)\}$$

The strongly regular graphs with $v \leq 36$ in the above list have been classified. If a regular two-graph has been classified, then we may classify the corresponding strongly regular graphs. When $v \geq 50$ the classifications of regular two-graphs are not known, except a regular two-graph on 276 vertices. Indeed, a regular two-graph on 276 vertices is unique [11]. Moreover, Goethals and Seidel [11] gave one strongly regular graph with parameters $(276, 140, 58, 84)$ in the switching class of the regular two-graph on 276 vertices. By Theorem 1.1, we can easily construct new strongly regular graphs with $(276, 140, 58, 84)$. Indeed, a 6-clique holds the conditions in Theorem 1.1. By the algebra software Magma, we can easily get the set of all 6-cliques. And, we make the set of 6-cliques up to transitiveness. It is easy to make the strongly regular graphs with the same parameters by switching with respect to the 6-clique. New strongly regular graphs are also applicable to this method. By repeating this method, we can efficiently get new examples. We found at least 100000 pairwise non-isomorphic strongly regular graphs with parameters $(276, 140, 58, 84)$. However, we have not succeeded the classification of strongly regular graphs with parameters $(276, 140, 58, 84)$, because there are too many induced subgraphs holding the conditions in Theorem 1.1. Since the disjoint union of spherical 1-designs is also a spherical 1-design, the disjoint union of 6 clique subgraphs also satisfy the condition in Theorem 1.1. We can guess 100000 strongly regular graphs are very small part of the classification.

Problem 6.1. The strongly regular graphs with parameters $(276, 140, 58, 84)$ are pseudogeometric $(5, 27, 3)$-graph. Is there a geometric strongly regular graph with there parameters? (please see [17] for the terminologies)

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